Improved Lower Bound, and Proof Barrier, for Constant Depth Algebraic Circuits

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We show that any product-depth Δ algebraic circuit for the Iterated Matrix Multiplication polynomial $\mathrm{IMM}_{n,d}$ (when $d = O(\log n / \log\log n)$) must be of size at least $n^{\Omega(d^{1/(\varphi^2)^\Delta})}$, where $\varphi = 1.618\ldots$ is the golden ratio. This improves the recent breakthrough result of Limaye, Srinivasan, and Tavenas (FOCS'21), who showed a super polynomial lower bound of the form $n^{\Omega(d^{1/4^\Delta})}$ for constant-depth circuits.

One crucial idea of the (LST21) result was to use set-multilinear polynomials where each set in the variables' underlying partition could be of different sizes. By picking the set sizes more carefully (depending on the depth we are working with), we first show that any product-depth Δ *set-multilinear* circuit for $\mathrm{IMM}_{n,d}$ (when $d = O(\log n)$) needs size at least $n^{\Omega(d^{1/\varphi^\Delta})}$. This improves the $n^{\Omega(d^{1/2^\Delta})}$ lower bound of (LST21). We then use their Hardness Escalation technique to lift this to general circuits.

We also show that these techniques cannot improve our lower bound significantly. For the *specific* two set sizes used in (LST21), they showed that their lower bound cannot be improved. We show that for any $d^{o(1)}$ set sizes (out of maximum possible d), the scope for improving our lower bound is minuscule. There exists a set-multilinear circuit that has product-depth Δ and size almost matching our lower bound such that the value of the measure used to prove the lower bound is maximum for this circuit. This results in a barrier to further improvement using the same measure.

CCS Concepts: • **Theory of computation → Algebraic complexity theory;**

Additional Key Words and Phrases: Polynomials, lower bounds, algebraic circuits, proof barrier, fibonacci numbers

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1 Introduction

An *Algebraic Circuit* is a natural model to compute multivariate polynomials over a field $\mathbb{F}$. It is a layered *directed acyclic graph* with leaves labeled by variables $x_1, \ldots, x_n$ or elements from $\mathbb{F}$. The

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internal nodes are alternating layers of either addition (+) or multiplication (×) gates. The circuit computes a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$ in a natural way: the + gates compute arbitrary $\mathbb{F}$-linear combination of their inputs and the × gates compute the product. Some associated *complexity measures* capture the computational complexity of computing the polynomial. The *size* of the circuit is the total number of nodes (edges) in the graph. The number of layers in the circuit is its *depth*. By *product-depth*, we mean the number of layers of multiplication gates (depth is roughly twice the product-depth). A subclass of circuits of particular importance is *Arithmetic Formulas*, whose underlying graph is a *tree*. For a general survey of the field of Algebraic Complexity Theory, see [3, 22, 32].

Valiant [35], in a very influential work, defined the classes VP and VNP, which can be considered the arithmetic analogs of P and NP. Much like in the Boolean world, the question of whether VP and VNP are the same is a central open problem in algebraic complexity theory. Though the best known lower bounds for general arithmetic circuits [2] ($\Omega(n \log n)$) and formulas [10] ($\Omega(n^2)$) fall far short of the super polynomial lower bounds that we hope to prove, such bounds are known for various restricted classes [24–26] (see also [4, 28] for excellent survey).

One of the most interesting restrictions is that of bounding the *depth* of circuits and formulas. When the depth is a constant, circuits and formulas are equivalent up to polynomial blow up in their size and hence we use them interchangeably in this article. Unlike the Boolean world, a very curious phenomenon of *depth reduction* occurs in arithmetic circuits [1, 8, 16, 33, 36]. Essentially, circuits of depth 3 and 4 are almost as powerful as general ones. More formally, any degree d polynomial f that has a size s circuit can also be computed by a depth 4 *homogeneous* circuit or a depth 3 (possibly non-homogeneous) circuit of size $s^{O(\sqrt{d})}$. Hence, proving an $n^{\omega(\sqrt{d})}$ lower bound on these special circuits is enough to separate VP from VNP. The extreme importance of bounded depth circuits has led to a large body of work proving lower bounds for these models and their variants [6, 7, 9, 11–15, 17, 18, 27, 30, 31].

1.1 The LST Breakthrough

In a remarkable recent work, Limaye, Srinivasan and Tavenas [21] proved the first superpolynomial lower bound for general constant-depth circuits. More precisely, they showed that the Iterated Matrix Multiplication polynomial $\mathrm{IMM}_{n,d}$ (where $d = o(\log n)$) has no product-depth Δ circuits of size $n^{d^{\exp(-O(\Delta))}}$. The polynomial $\mathrm{IMM}_{n,d}$ is defined on $N = dn^2$ variables. The variables are partitioned into d sets $X_1, \ldots, X_d$ of n^2 variables each (viewed as $n \times n$ matrices). The polynomial is defined as the $(1, 1)$th entry of the matrix product $X_1 X_2 \cdots X_d$. All monomials of the polynomial are of the same degree, so $\mathrm{IMM}_{n,d}$ is *homogeneous*. As the individual degree of any variable is at most 1, it is also *multilinear*. Moreover, every monomial has exactly one variable from each set $X_1, \ldots, X_d$, making the polynomial *set-multilinear*. For any $\Delta \leq \log d$, $\mathrm{IMM}_{n,d}$ has a set-multilinear circuit of product-depth Δ and size $n^{O(d^{1/\Delta})}$, obtained via basic recursion. No significantly better upper bounds are known, even if we allow general circuits. It makes sense to conjecture that this upper bound is tight (see [5] for limitations to improving the upper bound when the matrices X_i are 2×2).

An algebraic circuit is said to be *set-multilinear* if every node in the circuit computes a set-multilinear polynomial with respect to (a subset of) the variable sets. The lower bound of [21] proceeds by first transforming size s, product-depth Δ, general circuits computing a set-multilinear polynomial of degree d to *set-multilinear* algebraic circuits of product-depth 2Δ and size $\mathrm{poly}(s)d^{O(d)}$ (which is not huge if d is small). Hence, lower bounds on bounded depth set-multilinear circuits translate to bounded depth general circuit lower bounds, albeit with some loss. Finally, considering set-multilinear circuits with variables partitioned into sets of *different sizes* and crucially using this discrepancy of set sizes helps in obtaining strong set-multilinear lower bounds.

1.2 Our Results

In this work, we improve the lower bound for IMM against constant-depth circuits. We also exhibit barriers to further improving the bound using these techniques, which is important as this is the only known approach to achieve super polynomial lower bounds for general circuits of low depth.

For the rest of this article, let $\mu(\Delta) = 1/(F(\Delta) - 1)$, where $F(n) = \Theta(\varphi^n)$ is the nth Fibonacci number (starting with $F(0) = 1, F(1) = 2$) and $\varphi = (1 + \sqrt{5})/2 = 1.618\dots$ is the golden ratio.

Theorem 1.1 (General Circuit Lower Bound). *Fix a field $\mathbb{F}$ of characteristic 0 or characteristic $> d$. Let N, d, Δ be such that $d = O(\log N/\log\log N)$. Then, any product-depth Δ circuit computing $\mathrm{IMM}_{n,d}$ on $N = dn^2$ variables must have size at least $N^{\Omega\left(d^{\mu(2\Delta)}/\Delta\right)}$.*

Remark 1.2. Theorem 1.1 improves on the lower bound of $N^{\Omega(d^{1/(2^{2\Delta}-1)}/\Delta)}$ of [21] since $F(2\Delta) = \Theta(\varphi^{2\Delta}) \ll 2^{2\Delta}$.

To prove Theorem 1.1, we use the hardness escalation given by the following lemma from [21], which allows for the conversion of general circuits to set-multilinear ones without significant blow-up in size (provided the degree is small).

Lemma 1.3. *[21, Proposition 9] Let s, N, d, Δ be growing parameters with $s \geq Nd$. Assume that char$(\mathbb{F}) = 0$ or char$(\mathbb{F}) > d$. If C is a circuit of size at most s and product-depth at most Δ computing a set-multilinear polynomial P over the sets of variables $(X_1, \dots, X_d)$ (with $|X_i| \leq N$), then there is a* set-multilinear circuit *$\tilde{C}$ of size $d^{O(d)}\mathrm{poly}(s)$ and product-depth at most 2Δ computing the same polynomial P.*

The actual lower bound is for set-multilinear circuits.

Theorem 1.4 (Set-multilinear Circuit Lower Bound). *Let $d \leq (\log n)/4$. Any product-depth Δ* set-multilinear *circuit computing $\mathrm{IMM}_{n,d}$ must have size at least $n^{\Omega\left(d^{\mu(\Delta)}/\Delta\right)}$.*

Remark 1.5. Theorem 1.4 is an improvement over the $n^{\Omega(d^{1/(2^{\Delta}-1)}/\Delta)}$ bound of [21, Lemma 15]. The result holds over any field $\mathbb{F}$, same as [21]. The restriction on the characteristic in Theorem 1.1 comes from the conversion to set-multilinear circuits (Lemma 1.3). The difference between $\mu(2\Delta)$ in Theorem 1.1 and $\mu(\Delta)$ in Theorem 1.4 is also due to the doubling of product-depth during this conversion.

In a further recent work [34], Tavenas, Limaye and Srinivasan proved a product-depth Δ *set-multilinear* formula lower bound of $(\log n)^{\Omega(\Delta d^{1/\Delta})}$ for $\mathrm{IMM}_{n,d}$. There is no degree restriction, but in the small degree regime, the bound is much weaker than [21] and cannot be used for escalation. Improving on it, Kush and Saraf [19] showed a lower bound of $n^{\Omega(n^{1/\Delta}/\Delta)}$ for the size of product-depth Δ set-multilinear formulas computing an n^2-variate, degree n polynomial in VNP from the family of Nisan-Wigderson design-based polynomials. Unlike both [34] and [19], we are interested in the low-degree regime where set-multilinear lower bounds can be lifted, and our bounds will be for IMM (a polynomial in VP), making these works incomparable to ours. We now prove Theorem 1.1 à la [21, Corollary 4]:

Proof of Theorem 1.1. From Lemma 1.3 and Theorem 1.4, for a circuit of product-depth Δ and size s computing $\mathrm{IMM}_{n,d}$ we get that

$$d^{O(d)}\mathrm{poly}(s) \geq N^{\Omega\left(d^{\mu(2\Delta)}/2\Delta\right)}.$$

Since $d = O(\log N/\log\log N)$, it follows that $d^{O(d)} = N^{O(1)}$. Therefore,

$$\mathrm{poly}(s) \geq N^{\Omega\left(d^{\mu(2\Delta)}/2\Delta\right)}/d^{O(d)} \geq N^{\Omega\left(d^{\mu(2\Delta)}/4\Delta\right)}$$

implying the required lower bound on s and, thus, the theorem. □

Remark 1.6. Theorem 1.1 also holds when $d = O(\log N)$ and $\Delta \leq 1/4 \log_{\varphi} \log d$. This is because the above bound on Δ implies that $d^{\mu(2\Delta)}/\Delta \geq d^{\Omega(1/\varphi^{2\Delta})}/\Delta \geq d^{\Omega(1/\sqrt{\log d})}/\log\log d = \omega(\log d)$. Using this inequality together with the assumption $d = O(\log N)$, we get $d^{O(d)} = 2^{O(d\log d)} \leq 2^{o(\log N \cdot d^{\mu(2\Delta)}/\Delta)} = N^{o(d^{\mu(2\Delta)}/\Delta)}$ we can then proceed similarly to the proof of Theorem 1.1.

The hard polynomial for which we prove the set-multilinear lower bound is actually a word polynomial (Definition 2.1), which is a set-multilinear restriction of IMM (Lemma 2.2). Hence, the lower bound gets translated to $\mathrm{IMM}_{n,d}$. These word polynomials are set-multilinear with respect to $(X_1, \ldots, X_d)$, where each of the X_is could potentially be of different sizes.

For the two specific set sizes considered in [21], they also exhibit polynomials that match their lower bound. It still leaves open the question whether we can improve the lower bound by choosing some other set sizes. In Theorem 1.4, by choosing two set sizes that are distinctly different from the ones in [21], we show that it is indeed possible to improve their lower bound. It might then seem plausible that using many more set sizes could improve the bound further. We show that this is false for most cases. Suppose there are $\gamma \leq d$ different set sizes among the X_is. We show that there are set-multilinear polynomials that can be computed by product-depth Δ circuits having size roughly comparable to the size lower bound of Theorem 1.4, provided γ is not too large. Formally,

Theorem 1.7 (Barrier). *Let $s_1, \ldots, s_\gamma$ be positive integers. Fix sets $X_1, \ldots, X_d$ where for all i, $|X_i| \in \{s_1, \ldots, s_\gamma\}$. For any fixed positive integer Δ, there exist polynomials P_Δ and Q_Δ that are set-multilinear with respect to $X_1, \ldots, X_d$ such that P_Δ can be computed by product-depth Δ circuits of size $n^{O\left(\Delta\gamma d^{\mu(\Delta)}\right)}$ and Q_Δ can be computed by product-depth Δ circuits of size $n^{O\left(\Delta d^{\mu(\Delta-1)}+\gamma\right)}$. Moreover, P_Δ and Q_Δ maximize the measure used to prove lower bounds.*

Remark 1.8. The two different polynomials with slightly different sizes will imply barriers to improving the lower bound in different regimes of γ. Suppose that $\Delta = O(1)$ is small. When $\gamma = O(1)$, the size of P_Δ matches our lower bound, essentially implying its tightness. When γ is $d^{o(1)}$, the size of Q_Δ is only slightly larger than our lower bound (note $\mu(\Delta-1)$ vs $\mu(\Delta)$). Hence, even when multiple set sizes are used, the scope for improvement is tiny.

In an almost parallel work [20], Limaye, Srinivasan, and Tavenas show similar barrier results. They simplify the proof framework of [21] and characterize the lower bounds that can be proved via this technique using a combinatorial property, which they term *Tree Bias*. Their result works for any d set sizes, but the upper bound they obtain is weaker. More precisely, for any partition $(X_1, \ldots, X_d)$ of the input variables, they exhibit a set-multilinear polynomial that can be computed by product-depth Δ set-multilinear circuits of size $n^{d^{1/\Delta^{\Omega(\log\Delta)}}}$ while simultaneously maximizing the measure. These barrier results (Theorem 1.7 and results of [20]) suggest that new measures and techniques might be necessary if we are to improve the lower bounds significantly.

2 Preliminaries

For any positive integer n, we denote by $F(n)$ the nth Fibonacci number with $F(0) = 1$, $F(1) = 2$ and $F(n) = F(n-1) + F(n-2)$. The nearest integer to any real number r is denoted by $\lfloor r \rceil$. We follow the notation of [21] as much as possible for better readability.

We consider words that are tuples $(w_1, \ldots, w_d)$ of length d, where $2^{|w_i|}$ are integers. These words define the set sizes of the set-multilinear polynomials we will work with. Given a word w, let $\overline{X}(w)$ denote the tuple of sets of variables $(X_1(w), \ldots, X_d(w))$, where the size of each $X_i(w)$ is $2^{|w_i|}$. We denote the space of set-multilinear polynomials over $\overline{X}(w)$ by $\mathbb{F}_{sm}[\overline{X}(w)]$.

For a word w and any subset $S \subseteq [d]$, the sum of elements of w indexed by S is denoted by $w_S = \sum_{i \in S} w_i$. For all $t \leq d$, if it holds that $|w_{[t]}| \leq b$, then we call w "b-unbiased". Denote by $w_{|S}$ the sub-word indexed by S. The positive and negative indices of w are denoted $\mathcal{P}_w = \{i \mid w_i \geq 0\}$ and $\mathcal{N}_w = \{i \mid w_i < 0\}$, respectively, with the corresponding collections $\{X_i(w)\}_{i \in \mathcal{P}_w}$ and $\{X_i(w)\}_{i \in \mathcal{N}_w}$ being the positive and negative variable sets. We denote by $\mathcal{M}_w^{\mathcal{P}}$ (resp. $\mathcal{M}_w^{\mathcal{N}}$) the set of all set-multilinear monomials over the positive (resp. negative) variable sets.

The *partial derivative matrix* $\mathcal{M}_w(f)$ of f has rows indexed by $\mathcal{M}_w^{\mathcal{P}}$ and columns by $\mathcal{M}_w^{\mathcal{N}}$. The entry corresponding to row $m_+ \in \mathcal{M}_w^{\mathcal{P}}$ and $m_- \in \mathcal{M}_w^{\mathcal{N}}$ is the coefficient of the monomial $m_+ m_-$ in f. The complexity measure we use is the *relative rank*, same as [21]:

$$\operatorname{relrk}_w(f) := \frac{\operatorname{rank}(\mathcal{M}_w(f))}{\sqrt{|\mathcal{M}_w^{\mathcal{P}}| \cdot |\mathcal{M}_w^{\mathcal{N}}|}} = \frac{\operatorname{rank}(\mathcal{M}_w(f))}{2^{\frac{1}{2}\sum_{i \in [d]} |w_i|}} \leq 1.$$

The following properties of relrk_w will be useful (for proofs, see [21]).

(1) (Imbalance) For any $f \in \mathbb{F}_{sm}[\overline{X}(w)]$, $\operatorname{relrk}_w(f) \leq 2^{-|w_{[d]}|/2}$.
(2) (Sub-additivity) For any $f, g \in \mathbb{F}_{sm}[\overline{X}(w)]$, $\operatorname{relrk}_w(f+g) \leq \operatorname{relrk}_w(f) + \operatorname{relrk}_w(g)$.
(3) (Multiplicativity) Suppose $f = f_1 f_2 \cdots f_t$ where $f_i \in \mathbb{F}_{sm}[\overline{X}(w_{|S_i})]$ and $(S_1, \ldots, S_t)$ is a partition of $[d]$. Then, $\operatorname{relrk}_w(f) = \operatorname{relrk}_w(f_1 f_2 \cdots f_t) = \prod_{i \in [t]} \operatorname{relrk}_{w_{|S_i}}(f_i)$.

We now define the hard polynomials for which we prove lower bounds. For any monomial $m \in \mathbb{F}_{sm}[\overline{X}(w)]$, let $m_+ \in \mathcal{M}_w^{\mathcal{P}}$ and $m_- \in \mathcal{M}_w^{\mathcal{N}}$ be its "positive" and "negative" parts. As $|X_i| = 2^{|w_i|}$, the variables of X_i can be indexed using Boolean strings of length $|w_i|$. This gives a way to associate a Boolean string with any monomial. Let $\sigma(m_+)$ and $\sigma(m_-)$ be the strings associated with m_+ and m_-, respectively. We write $\sigma(m_+) \sim \sigma(m_-)$ if one is a prefix of the other.

Definition 2.1. [21, Word polynomials] Let w be any word. The polynomial P_w is defined as the sum of all monomials $m \in \mathbb{F}_{sm}[\overline{X}(w)]$ such that $\sigma(m_+) \sim \sigma(m_-)$.

The matrices $M_w(P_w)$ have full rank (equal to either the number of rows or columns, whichever is smaller), and hence $\operatorname{relrk}_w(P_w) = 2^{-|w_{[d]}|/2}$. We note (without proof) that these polynomials can be obtained as *set-multilinear* restrictions of $\mathrm{IMM}_{n,d}$.

Lemma 2.2. *[21, Lemma 8] Let w be any b-unbiased word. If there is a set-multilinear circuit computing* $\mathrm{IMM}_{2^b,d}$ *of size s and product-depth Δ, then there is also a set-multilinear circuit of size s and product-depth Δ computing the polynomial $P_w \in \mathbb{F}_{sm}[\overline{X}(w)]$. Moreover,* $\operatorname{relrk}_w(P_w) \geq 2^{-b/2}$.

3 Proof Outline

From the discussion in Section 1 and Lemmas 2.2 and 1.3, to prove general circuit lower bounds, it suffices to prove that there is a word polynomial of high rank that needs large set-multilinear formulas. For a word (and hence set sizes) of our choice, we show that relrk_w is small for set-multilinear formulas of a certain size.

Let k be an integer close to $\log_2 n$. In [21], the authors choose the positive entries of the word w as an integer close to $k/\sqrt{2}$ and the negative entries as $-k$. Evidently, these entries are independent of the product-depth Δ. In this article, we take the positive entries as $(1 - p/q)k$ and the negative entries as $-k$, where p and q are suitable integers dependent on Δ. This *depth-dependent* construction of the word enables us to improve the lower bound. We demonstrate the high-level proof strategy of the lower bound for the case of product-depth 3.

3.1 Proof Overview of Theorem 1.4 for $\Delta = 3$

Define $G(i) = 1/\mu(i) = F(i) - 1$ for all i and let $\lambda = \lfloor d^{1/G(3)} \rfloor$. Consider a set-multilinear formula C of product-depth 3 and let v be a gate in it. Suppose that the subformula $C^{(v)}$ rooted at v has product-depth $\delta \leq 3$, size s and degree $\geq \lambda^{G(\delta)}/2$. We will prove that $\text{relrk}_w(C^{(v)}) \leq s2^{-k\lambda/48}$ by induction on δ. This will give us the desired upper bound of the form $s2^{-k\lambda/48} = sn^{-\Omega(d^{\mu(3)})}$ on the relative rank of the whole formula when v is taken to be the output gate. Write $C^{(v)} = C_1 + \cdots + C_t$, where each C_i is a subformula of size s_i rooted at a product gate. Because of the subadditivity of relrk_w, it suffices to show that $\text{relrk}_w(C_i) \leq s_i 2^{-k\lambda/48}$ for all i.

Base case: If $\delta = 1$, then C_i is a product of linear forms. Thus, it has a rank of 1 and a low relative rank.

Induction step: $\delta \in \{2, 3\}$. Write $C_i = C_{i,1} \ldots C_{i,t_i}$ where each $C_{i,j}$ is a subformula of product-depth $\delta - 1$. If any $C_{i,j}$ has degree $\geq \lambda^{G(\delta-1)}/2$, then by the induction hypothesis, the relative rank of $C_{i,j}$ and hence C_i will have the desired upper bound, and we are done.

Otherwise each $C_{i,j}$ has degree $D_{ij} < \lambda^{G(\delta-1)}/2$. As the formula is set-multilinear, there is a collection of variable sets $(X_l)_{l \in S_j}$ with respect to which $C_{i,j}$ is set-multilinear. For $j \in [t_i]$, let a_{ij} be the number of positive indices in S_j that is, the number of positive sets in the collection $(X_l)_{l \in S_j}$. Then the number of negative indices is $(D_{ij} - a_{ij})$.

We consider two cases: if $a_{ij} \leq D_{ij}/3$, then $w_{S_j} \leq (D_{ij}/3) \cdot (1 - p/q)k + (2D_{ij}/3) \cdot (-k) \leq -D_{ij}k/3$. Otherwise $a_{ij} > D_{ij}/3$ and if we can prove that $|w_{S_j}| \geq a_{ij}k/(4\lambda^{G(\delta)-1})$, then in both of the above cases, we would have $|w_{S_j}| \geq D_{ij}k/(12\lambda^{G(\delta)-1})$. By the multiplicativity and imbalance property of relrk_w, it would follow that $\text{relrk}_w(C_i) \leq 2^{\sum_{j=1}^{t_i} -\frac{1}{2}|w_{S_j}|} \leq 2^{-k\lambda/48}$ and we would be done. Thus, we now only have to show that $|w_{S_j}| \geq a_{ij}k/(4\lambda^{G(\delta)-1})$. We have

$$|w_{S_j}| = \left|a_{ij}(1 - p/q) - (D_{ij} - a_{ij})\right| k.$$

Notice that $|w_{S_j}|/k$ is the distance of $a_{ij}p/q$ from some integer, so it must be at least the minimum of $\{a_{ij}p/q\}$ and $1 - \{a_{ij}p/q\}$, where $\{.\}$ denotes the fractional part. The number $a_{ij}p/q$ being rational, has a fractional part $\zeta = (a_{ij}p \bmod q)/q$, and hence it comes down to solving the following system of inequalities:

$$\min(\zeta,\ 1 - \zeta) \geq a_{ij}/(4\lambda^{G(\delta)-1}) \text{ for } \delta \in \{2, 3\} \text{ when } a_{ij} \leq D_{ij} < \lambda^{G(\delta-1)}/2.$$

Assign $p = \lambda$, $q = \lambda^2 + 1$. The $\delta = 2$ case is clearly satisfied as $(a_{ij}\lambda \mod (\lambda^2 + 1)) = a_{ij}\lambda$ when $0 \leq a_{ij} \leq \lambda/2$.

Consider the case of $\delta = 3$ and $a_{ij} < \lambda^2/2$. Write $a_{ij} = y_1\lambda + y_0$ for integers $y_1 = \lfloor a_{ij}/\lambda \rfloor < \lambda/2$ and $y_0 \leq \lambda - 1$. Thus, $a_{ij}\lambda \equiv -y_1 + y_0\lambda \mod (\lambda^2 + 1)$. Through some case analysis, one can show that $\min(|y_0\lambda - y_1|,\ \lambda^2 + 1 - |y_0\lambda - y_1|) \geq y_1$ which immediately implies the inequality for the $\delta = 3$ case as $y_1 = \lfloor a_{ij}/\lambda \rfloor \geq a_{ij}/(2\lambda)$.

We can attempt to extend this proof technique to product-depth 4 as follows:
We would similarly want to express a_{ij} as $a_{ij} = y_2\lambda^2 + y_1\lambda + y_0$ for integers $y_2 = \lfloor a_{ij}/\lambda^2 \rfloor$, $y_0 \leq \lambda - 1$ and $y_1 \leq \lambda - 1$. Ideally, we would want that for some $q \approx \lambda^4$,

$$p\lambda^2 \equiv 1 \bmod q,\ p\lambda \equiv -\lambda^2 \bmod q \text{ and } p \equiv \lambda^3 \bmod q$$

so that $a_{ij}p \equiv y_2 - y_1\lambda^2 + y_0\lambda^3 \bmod q$ and then we can carry out a similar analysis as in the $\Delta = 3$ case. But this is impossible since multiplying the second congruence equation by λ gives $p\lambda^2 \equiv -\lambda^3 \bmod q$, which contradicts the first congruence equation. So we decide to express a_{ij} as

$a_{ij} = y_2b_2 + y_1b_1 + y_0b_0$, where b_2, b_1, b_0 are close to $\lambda^2, \lambda, 1$, respectively, instead of being precisely equal to these powers of λ. Then we choose $c_2 \approx 1, c_1 \approx -\lambda^2, c_0 \approx \lambda^3$ and we assign values to p and q such that

$$pb_2 \equiv c_2 \bmod q,\ pb_1 \equiv c_1 \bmod q \text{ and } pb_0 \equiv c_0 \bmod q.$$

It is easy to verify that all these conditions are satisfied if we define
$b_0 = 1, b_1 = \lambda, b_2 = b_1(\lambda - 1) + b_0$; $\quad c_2 = 1, c_1 = -\lambda^2, c_0 = c_2 - c_1(\lambda - 1)$;
$p = c_0$ and $q = pb_1 - c_1$.
This inspired our construction of the sequences $\{b_m\}$ and $\{c_m\}$ for general product-depth Δ.

3.2 Proof Overview of Theorem 1.7

As previously mentioned, we would like to find a family of polynomials for which our lower bound is tight. All the same, we want to maintain a high relative rank of these polynomials. If we can achieve this and find the appropriate small-sized formulas for the said polynomials, we will have that the lower bound cannot be improved using the relative rank measure.

The polynomial P we define will be a close variant of the word polynomials from before. This will ensure that the partial derivative matrix has the maximum possible rank for a matrix of its dimension. From the Imbalance property, the relative rank we obtain is $2^{-|w_{[d]}|/2}$ where we have ensured that $w_{[d]}$ is small. We want to construct the formula F for P with a nice inductive structure. We also want the polynomials computed by the subformulas of F to have a high relative rank. This will help us construct a formula from its sub-formulas while maintaining a high relative rank.

Suppose a subformula F' of F is set multilinear with respect to a subtuple $\mathcal{T}$ of the sets of variables $\overline{X}(w)$. Let these sets in $\mathcal{T}$ be indexed by a set $S_{\mathcal{T}} \subseteq [d]$. As we would like high relative rank of F', the Imbalance property again suggests that $|w_{S_{\mathcal{T}}}|$ be small. And we desire this of every subformula, their subformulas, and so on. So roughly, we want a way to partition our initial index set $[d]$ into some number of index sets $S_1, \ldots, S_r$ such that each $|w_{S_i}|$ is small. Suppose we can then create subformulas of rank $2^{-|w_{S_i}|/2}$. We will have to roughly add $2^{\sum_i |w_{S_i}|}$ many of them to get a polynomial of high relative rank. So, to control the size of the formula, we would like $\sum_i |w_{S_i}|$ to be small as well.

In their Depth Hierarchy section, [21] use Dirichlet's approximation principle [29] to pick these nice index sets $\{S_i\}$. Their procedure only works for the particular two variable-set sizes they choose. We extend this to any two set sizes in Claim 5.6. Interestingly, we do not use Dirichlet's approximation to pick the index sets but rather to obtain a lower bound on the size of the sets we eventually pick. We think of picking sets as an investment process: when we pick a set S, we buy the $|S|$ elements in it for a cost of $|w_S|$. Hence, the cost per element is $|w_S|/|S|$. At each product-depth, we are only allowed to pick sets of size under a certain threshold, and we pick the ones with the lowest cost per element. It turns out that this lowest cost decreases exponentially as the depth increases, which helps us build a small formula. The decrease is captured by the Fibonacci numbers and is the reason why they emerge in our lower bound and upper bound.

Making these ideas precise requires substantial notation, and we postpone further discussion to Section 5.

4 The Lower Bound: Proof of Theorem 1.4

Fix the product-depth Δ for which we want to prove the set-multilinear formula lower bound. Define $G(i) := F(i) - 1$ for all i and $\lambda = \lfloor d^{1/G(\Delta)} \rfloor$. We can assume that $\lambda \geq 3$ because otherwise $d^{\mu(\Delta)} < 3$, and in that case, the lower bound is trivial. The lower bound we aim to prove is $n^{\Omega(d^{1/G(\Delta)})}$.

4.1 The Sequences $\{b_m\}$, $\{c_m\}$ and the Integers p, q

We first define the sequences of integers $\{b_m\}$ and $\{c_m\}$ mentioned in the proof overview:

Let $r_m := \lambda^{G(m+1)-G(m)} - 1$ for $0 \leq m \leq \Delta - 2$.

Define

$$b_0 := 1, \quad b_1 := \lambda \text{ and } b_m := b_{m-2} + r_{m-1} b_{m-1} \text{ for } 2 \leq m \leq \Delta - 2.$$

Define

$$c_{\Delta-2} := (-1)^{\Delta-2}, \qquad c_{\Delta-3} := (-1)^{\Delta-3} \lambda^{G(\Delta-1)-G(\Delta-2)} \text{ and}$$
$$c_m := (-1)^m (|c_{m+2}| + r_{m+1} |c_{m+1}|) \text{ for } \Delta - 4 \geq m \geq 0.$$

Note that the sign parity of c_m is $(-1)^m$ for all m.

Thus, $c_{m-2} = (-1)^{m-2}(|c_m| + r_{m-1}|c_{m-1}|) = c_m - r_{m-1}c_{m-1}$ which implies

$$c_m = c_{m-2} + r_{m-1} c_{m-1} \text{ for } 2 \leq m \leq \Delta - 2.$$

Observe that as m increases, b_m increases and $|c_m|$ decreases.

Define

$$p := c_0 \text{ and } q := pb_1 - c_1 = c_0(r_0 + 1) - c_1.$$

By defining the integers p and q this way, we have ensured that $pb_0 \equiv c_0 \bmod q$ and $pb_1 \equiv c_1 \bmod q$. Hence from the relations $b_m = b_{m-2} + r_{m-1}b_{m-1}$ and $c_m = c_{m-2} + r_{m-1}c_{m-1}$, it inductively follows that

$$pb_m \equiv c_m \bmod q \qquad \text{for } 0 \leq m \leq \Delta - 2. \tag{4.1}$$

4.2 Bounds on the Values of b_m and $|c_m|$

Each b_m is close to $\lambda^{G(m)}$ and each $|c_m|$ is close to $\lambda^{G(\Delta-1)-G(m+1)}$:

Lemma 4.1. *Let $\lambda \geq 3$ be as defined in Section 4. Then for $0 \leq m \leq \Delta - 2$, $\frac{\lambda^{G(m)}}{2} \leq b_m \leq \lambda^{G(m)}$, and $\frac{\lambda^{G(\Delta-1)-G(m+1)}}{2} \leq |c_m| \leq \lambda^{G(\Delta-1)-G(m+1)}$.*

To prove these bounds, we use a generalized version of the well-known Bernoulli's inequality [23, Section 2.4]:

Claim 4.2 (Bernoulli's Inequality). *Let $x_1, \ldots, x_r$ be real numbers all greater than -1 and all with the same sign. Then,*

$$(1 + x_1)(1 + x_2) \ldots (1 + x_r) \geq 1 + x_1 + \cdots + x_r\, .$$

Proof of Lemma 4.1. Clearly, b_m satisfies the bounds when $m = 0$ or 1. For $m \geq 2$,

$$\begin{aligned} b_m &= (\lambda^{G(m)-G(m-1)} - 1) b_{m-1} + b_{m-2} \\ &\leq \lambda^{G(m)-G(m-1)} b_{m-1} \\ &\leq \lambda^{G(m)-G(m-1)} . \lambda^{G(m-1)-G(m-2)} \ldots \lambda^{G(2)-G(1)} b_1 \\ &= \lambda^{G(m)}. \end{aligned}$$

$$
\begin{aligned}
b_m &= (\lambda^{G(m)-G(m-1)} - 1)b_{m-1} + b_{m-2} \\
&\geq (\lambda^{G(m)-G(m-1)} - 1)b_{m-1} \\
&\geq (\lambda^{G(m)-G(m-1)} - 1).(\lambda^{G(m-1)-G(m-2)} - 1)\dots(\lambda^{G(2)-G(1)} - 1)b_1 \\
&= \lambda^{G(m)-G(1)}b_1.\left(1 - \frac{1}{\lambda^{G(m)-G(m-1)}}\right)\left(1 - \frac{1}{\lambda^{G(m-1)-G(m-2)}}\right)\dots\left(1 - \frac{1}{\lambda^{G(2)-G(1)}}\right) \\
&\geq \lambda^{G(m)}.\left(1 - \frac{1}{\lambda^{G(m)-G(m-1)}} - \frac{1}{\lambda^{G(m-1)-G(m-2)}} - \dots - \frac{1}{\lambda^{G(2)-G(1)}}\right) && \text{[By Claim 4.2]} \\
&\geq \lambda^{G(m)}.\left(1 - \frac{1}{\lambda^{m-1}} - \frac{1}{\lambda^{m-2}} - \dots - \frac{1}{\lambda}\right) && \text{since } G(i) \geq i \\
&= \lambda^{G(m)}.\left(1 - \frac{1}{\lambda - 1}\left(1 - \frac{1}{\lambda^{m-1}}\right)\right) \geq \frac{\lambda^{G(m)}}{2}.
\end{aligned}
$$

Clearly, $|c_m|$ satisfies the bounds when $m = \Delta - 2$ or $\Delta - 3$. For $m \leq \Delta - 4$,

$$
\begin{aligned}
|c_m| &= (\lambda^{G(m+2)-G(m+1)} - 1)|c_{m+1}| + |c_{m+2}| \\
&\leq \lambda^{G(m+2)-G(m+1)}|c_{m+1}| \\
&\leq \lambda^{G(m+2)-G(m+1)} \cdot \lambda^{G(m+3)-G(m+2)} \dots \lambda^{G(\Delta-2)-G(\Delta-3)}|c_{\Delta-3}| \\
&= \lambda^{G(\Delta-2)-G(m+1)} \cdot \lambda^{G(\Delta-1)-G(\Delta-2)} = \lambda^{G(\Delta-1)-G(m+1)}.
\end{aligned}
$$

$$
\begin{aligned}
|c_m| &= (\lambda^{G(m+2)-G(m+1)} - 1)|c_{m+1}| + |c_{m+2}| \\
&\geq (\lambda^{G(m+2)-G(m+1)} - 1)|c_{m+1}| \\
&\geq (\lambda^{G(m+2)-G(m+1)} - 1) \cdot (\lambda^{G(m+3)-G(m+2)} - 1)\dots(\lambda^{G(\Delta-2)-G(\Delta-3)} - 1)|c_{\Delta-3}| \\
&= \lambda^{G(\Delta-2)-G(m+1)}|c_{\Delta-3}| \cdot \left(1 - \frac{1}{\lambda^{G(m+2)-G(m+1)}}\right)\left(1 - \frac{1}{\lambda^{G(m+3)-G(m+2)}}\right)\dots \\
&\qquad \dots\left(1 - \frac{1}{\lambda^{G(\Delta-2)-G(\Delta-3)}}\right) \\
&\geq \lambda^{G(\Delta-2)-G(m+1)}|c_{\Delta-3}|.\left(1 - \frac{1}{\lambda^{G(m+2)-G(m+1)}} - \dots - \frac{1}{\lambda^{G(\Delta-2)-G(\Delta-3)}}\right) \\
&\qquad \text{[By Claim 4.2]} \\
&\geq \lambda^{G(\Delta-2)-G(m+1)}|c_{\Delta-3}|.\left(1 - \frac{1}{\lambda^{m+1}} - \frac{1}{\lambda^{m+2}} - \dots - \frac{1}{\lambda^{\Delta-3}}\right) \\
&= \lambda^{G(\Delta-1)-G(m+1)}.\left(1 - \frac{1}{\lambda^m(\lambda - 1)}\left(1 - \frac{1}{\lambda^{\Delta-3-m}}\right)\right) \geq \frac{\lambda^{G(\Delta-1)-G(m+1)}}{2}.
\end{aligned}
$$

□

4.3 Constructing the Word and Proving the Lower Bound

Define $\alpha = 1 - p/q$. As $\frac{p}{q} \leq \frac{c_0}{c_0(r_0+1)} = 1/\lambda$, we have $\alpha \geq 1/2$. Since $q = c_0\lambda - c_1$, it implies that

$$q \leq |c_0|\lambda + |c_1| \leq 2\lambda^{G(\Delta-1)} \leq d < \lfloor \log_2 n \rfloor / 2$$

where the second inequality follows from the upper bound on each $|c_m|$ in Lemma 4.1. Therefore, there exists a multiple of q in the interval $[\frac{\lfloor \log_2 n \rfloor}{2}, \lfloor \log_2 n \rfloor]$. Let k be this multiple of q.
Then αk is an integer. We can construct a word w over the alphabet $\{\alpha k, -k\}$ such that w is k-unbiased. This can be done using induction: if $|w_{[i]}| \leq 0$, set $w_{i+1} = \alpha k$, otherwise set $w_{i+1} = -k$.

With these definitions in place, we can prove Theorem 1.4. Assume the following lemma:

Lemma 4.3. *Let $\delta \leq \Delta$ be an integer and α, k be as defined above. Let w be any word of length d over the alphabet $\{\alpha k, -k\}$. Then any set-multilinear formula C of product-depth δ, degree $D \geq \lambda^{G(\delta)}/8$ and size at most s satisfies*

$$\mathrm{relrk}_w(C) \leq s 2^{-k\lambda/256}.$$

Proof of Theorem 1.4. Note that by Lemma 2.2, there exists a set-multilinear projection P_w of $\mathrm{IMM}_{2^k,d}$ such that $\mathrm{relrk}_w(P_w) \geq 2^{-k}$. Consider a set-multilinear circuit of size s and product-depth Δ computing $\mathrm{IMM}_{n,d}$. We can expand it to a set-multilinear formula of size at most $s^{2\Delta}$ which computes the same polynomial. Hence, we will also have a set-multilinear formula of size at most $s^{2\Delta}$ computing P_w. As $d \geq \lambda^{G(\Delta)}/8$, taking the particular case of $\delta = \Delta$ in Lemma 4.3, we obtain $\mathrm{relrk}_w(P_w) \leq s^{2\Delta} 2^{-k\lambda/256}$. This gives the desired lower bound

$$s^{2\Delta} \geq 2^{-k} 2^{k\lambda/256} \geq \left(\sqrt{\frac{n}{2}}\right)^{\lfloor d^{1/G(\Delta)} \rfloor/256} /n = n^{\Omega(d^{\mu(\Delta)})}. \qquad \square$$

Proof of Lemma 4.3. We proceed by induction on δ. We can write $C = C_1 + \cdots + C_t$, where each C_i is a subformula of size s_i rooted at a product gate. Because of the subadditivity of relrk_w, it suffices to show that

$$\mathrm{relrk}_w(C_i) \leq s_i 2^{-k\lambda/256} \text{ for all } i.$$

Base case: C has product-depth $\delta = 1$ and degree $D \geq \lambda/8$. Then C_i is a product of linear forms. If L is linear form on some variable set $X(w_j)$, then $\mathrm{relrk}_w(L) \leq 2^{-|w_j|/2} \leq 2^{-k/4}$. Therefore, by the multiplicativity of relrk_w,

$$\mathrm{relrk}_w(C_i) \leq 2^{-kD/4} \leq 2^{-k\lambda/32}.$$

Induction hypothesis: Assume that the lemma is true for all product-depths $\leq \delta - 1$.
Induction step: Let C be a formula of product-depth δ and degree $D \geq \lambda^{G(\delta)}/8$. We can write $C_i = C_{i,1} \ldots C_{i,t_i}$, where each $C_{i,j}$ is a subformula of product-depth $\delta - 1$.

If C_i has a factor, say $C_{i,1}$, of degree $\geq \lambda^{G(\delta-1)}/8$, then by induction hypothesis,

$$\mathrm{relrk}_w(C_i) \leq \mathrm{relrk}_w(C_{i,1}) \leq s_i 2^{-k\lambda/256}.$$

Otherwise every factor of C_i has degree $< \lambda^{G(\delta-1)}/8$. Let $C_i = C_{i,1} \ldots C_{i,t_i}$, where each $C_{i,j}$ has degree $D_{ij} < \lambda^{G(\delta-1)}/8$. If C_i is set-multilinear with respect to $(X_l)_{l \in S}$, then let $(S_1, \ldots, S_{t_i})$ be the partition of S such that each $C_{i,j}$ is set-multilinear with respect to $(X_l)_{l \in S_j}$.

For $j \in [t_i]$, let a_{ij} be the number of positive indices in S_j. We have two cases: If $a_{ij} \leq D_{ij}/2$, then

$$w_{S_j} \leq \frac{D_{ij}}{2} \cdot \alpha k + \frac{D_{ij}}{2} \cdot (-k) = -\frac{D_{ij} p}{2q} k \leq -\frac{D_{ij} k}{4\lambda},$$

where the last inequality follows from $\frac{p}{q} \geq \frac{c_0}{2c_0(r_0+1)} = \frac{1}{2\lambda}$. The other case is $a_{ij} > D_{ij}/2$. If we can prove that $|w_{S_j}| \geq a_{ij} k/(8\lambda^{G(\delta)-1})$, then in both of the above cases, we would have $|w_{S_j}| \geq D_{ij} k/(16\lambda^{G(\delta)-1})$. By the multiplicativity and imbalance property of relrk_w and the assumption $D \geq \lambda^{G(\delta)}/8$, it would follow that

$$\mathrm{relrk}_w(C_i) \leq \prod_{j=1}^{t_i} 2^{-\frac{1}{2}|w_{S_j}|} \leq 2^{-\sum_{j=1}^{t_i} D_{ij} k/(32\lambda^{G(\delta)-1})} = 2^{-Dk/(32\lambda^{G(\delta)-1})} \leq 2^{-k\lambda/256}$$

and we would be done. Thus, we now only have to show that $|w_{S_j}| \geq a_{ij} k/(8\lambda^{G(\delta)-1})$.

$$\begin{aligned} |w_{S_j}| &= \left|a_{ij} \cdot \alpha k + (D_{ij} - a_{ij}) \cdot (-k)\right| = \left|a_{ij} \frac{p}{q} - (2a_{ij} - D_{ij})\right| k && \text{as } \alpha = 1 - p/q \\ &\geq \left|\frac{a_{ij} p}{q} - \left\lfloor \frac{a_{ij} p}{q} \right\rceil\right| k, && \text{where } \lfloor . \rceil \text{ denotes the nearest integer.} \end{aligned}$$

The fractional part of $\frac{a_{ij}p}{q}$ is $\frac{a_{ij}p \bmod q}{q}$. Hence to prove that $|w_{S_j}| \geq a_{ij}k/(8\lambda^{G(\delta)-1})$, it is enough to verify that the following inequality is satisfied:

$$\min\left(\frac{a_{ij}p \bmod q}{q}, 1 - \frac{a_{ij}p \bmod q}{q}\right) \geq \frac{a_{ij}}{8\lambda^{G(\delta)-1}}. \tag{4.2}$$

Showing that the p, q we defined satisfy the inequality in Equation (4.2):

We will first find what we call the base $(b_0, \ldots, b_{\Delta-2})$ representation of the number a_{ij}. For $0 \leq m \leq \Delta - 2$, inductively define y_m to be the integer quotient when $(a_{ij} - \sum_{m'=m+1}^{\Delta-2} b_{m'}y_{m'})$ is divided by b_m. Then we can express a_{ij} as $a_{ij} = \sum_{m=0}^{\Delta-2} b_m y_m$. Since $b_m \geq \lambda^{G(m)}/2$ for all m (Lemma 4.1) and $a_{ij} \leq D_{ij} < \lambda^{G(\delta-1)}/8$, we have the following bounds on the values of y_m:

$$y_m = 0 \text{ for } m \geq \delta - 1, \tag{4.3}$$

$$y_{\delta-2} = \left\lfloor \frac{a_{ij}}{b_{\delta-2}} \right\rfloor < \frac{\frac{\lambda^{G(\delta-1)}}{8}}{\frac{\lambda^{G(\delta-2)}}{2}} \leq \frac{\lambda^{G(\delta-1)-G(\delta-2)} - 1}{2} = \frac{r_{\delta-2}}{2}, \tag{4.4}$$

$$y_m \leq \left\lfloor \frac{b_{m+1}-1}{b_m} \right\rfloor = r_m \text{ for } m < \delta - 2. \tag{4.5}$$

By Equation (4.1), $a_{ij}p \equiv \sum_{m=0}^{\Delta-2} c_m y_m \bmod q$.

Define f to be the highest index such that $y_f \geq 1$ (by Equation (4.3), $f \leq \delta - 2$) and e to be the smallest index such that $y_e \geq 1$. Then $a_{ij}p \equiv \sum_{m=e}^{f} c_m y_m \bmod q$. Therefore,

$$\min\left(\frac{a_{ij}p \bmod q}{q}, 1 - \frac{a_{ij}p \bmod q}{q}\right) = \min\left(\left|\sum_{m=e}^{f} c_m y_m\right|/q,\ 1 - \left|\sum_{m=e}^{f} c_m y_m\right|/q\right) \tag{4.6}$$

if $|\sum_{m=e}^{f} c_m y_m|/q \leq 1$, which is true by the upper bound in the following claim (See the end of this section for the proof):

Claim 4.4. *If $0 \leq y_m \leq r_m$ for all m and $y_e \geq 1$, then $|c_{2\lfloor(f-e+1)/2\rfloor+e}| \leq |\sum_{m=e}^{f} c_m y_m| < q - c_0$ and the sign parity of $\sum_{m=e}^{f} c_m y_m$ is $(-1)^e$.*

Now, we prove a lower bound on the RHS of Equation (4.6). We have three cases:

— If $f < \delta - 2$, then $y_{\delta-2} = 0$ that is, $a_{ij} < b_{\delta-2}$. By Claim 4.4, we have

$$\min\left(\left|\sum_{m=e}^{f} c_m y_m\right|/q,\ 1 - \left|\sum_{m=e}^{f} c_m y_m\right|/q\right) \geq \frac{1}{q}\min(|c_{2\lfloor(f-e+1)/2\rfloor+e}|, c_0) \geq \frac{|c_{\delta-2}|}{q} > \left|\frac{c_{\delta-2}a_{ij}}{b_{\delta-2}q}\right|$$

where the second inequality follows from $2\lfloor(f-e+1)/2\rfloor + e \leq f + 1 \leq \delta - 2$.

— If $e = f = \delta - 2$, then $a_{ij} = b_{\delta-2}y_{\delta-2}$. Hence,

$$\min\left(\left|\sum_{m=e}^{f} c_m y_m\right|/q,\ 1 - \left|\sum_{m=e}^{f} c_m y_m\right|/q\right) = \frac{|c_{\delta-2}|y_{\delta-2}}{q} \qquad \text{since } |c_{\delta-2}|y_{\delta-2} \leq \frac{|c_{\delta-2}|r_{\delta-2}}{2} \leq \frac{c_0 r_0}{2} < q/2$$

$$= \left|\frac{c_{\delta-2}a_{ij}}{b_{\delta-2}q}\right|.$$

— If $e < f = \delta - 2$, then use the bounds of Lemma 4.1 to see that

$$|c_{\delta-3}| > \frac{\lambda^{G(\Delta-1)-G(\delta-2)}}{2} > \frac{\lambda^{G(\Delta-1)-G(\delta-1)}}{\lambda^{G(\delta-2)}/2} \cdot \frac{\lambda^{G(\delta-1)}}{8} > \frac{|c_{\delta-2}|}{b_{\delta-2}}a_{ij}.$$

Thus, Claim 4.4 implies that

$$1-\left|\sum_{m=e}^{f} c_m y_m\right|/q > \frac{c_0}{q} \geq \frac{c_{\delta-3}}{q} > \left|\frac{c_{\delta-2}a_{ij}}{b_{\delta-2}q}\right|.$$

If e and f have the same parity, then

$$\begin{aligned}\left|\sum_{m=e}^{f} c_m y_m\right|/q &\geq \frac{1}{q}|c_f y_f| && \text{since } \sum_{m=e}^{f-1} c_m y_m \text{ has same sign parity as } c_f y_f \text{ by Claim 4.4}\\ &= \frac{1}{q}\left|c_{\delta-2}\left\lfloor\frac{a_{ij}}{b_{\delta-2}}\right\rfloor\right| \geq \left|\frac{c_{\delta-2}a_{ij}}{2b_{\delta-2}q}\right|.\end{aligned}$$

If e and f have different parity, then

$$\begin{aligned}\left|\sum_{m=e}^{f} c_m y_m\right|/q &\geq \frac{1}{q}\left(\left|\sum_{m=e}^{f-1} c_m y_m\right| - |c_f y_f|\right)\\ &\geq \frac{1}{q}\left(|c_{2\lfloor(f-e)/2\rfloor+e}| - \frac{|c_f|r_f}{2}\right) && \text{by Claim 4.4}\\ &> \frac{|c_{f-1}|}{2q} && \text{since } |c_{2\lfloor(f-e)/2\rfloor+e}| = |c_{f-1}| > |c_f|r_f\\ &> \left|\frac{c_{\delta-2}a_{ij}}{2b_{\delta-2}q}\right| && \text{since } f-1=\delta-3.\end{aligned}$$

Thus in all three cases, $\min(|\sum_{m=e}^{f} c_m y_m|/q,\ 1-|\sum_{m=e}^{f} c_m y_m|/q) \geq |\frac{c_{\delta-2}a_{ij}}{2b_{\delta-2}q}|$.

By Lemma 4.1, we have

$$|c_{\delta-2}| \geq \lambda^{G(\Delta-1)-G(\delta-1)}/2,\ b_{\delta-2} \leq \lambda^{G(\delta-2)},\ q \leq |c_0|\lambda + |c_1| \leq 2\lambda^{G(\Delta-1)}.$$

Hence $\min(|\sum_{m=e}^{f} c_m y_m|/q,\ 1-|\sum_{m=e}^{f} c_m y_m|/q) \geq \frac{a_{ij}}{8\lambda^{G(\delta-1)+G(\delta-2)}} = \frac{a_{ij}}{8\lambda^{G(\delta)-1}}$ which together with Equation (4.6) implies Equation (4.2). □

We show the technical Claim 4.4 used in the above proof.

Proof of Claim 4.4. Define $c_{-1} = |c_0|r_0 + |c_1|$. Then from the definition of c_m, for all $m \geq 0$, $|c_m|r_m = |c_{m-1}| - |c_{m+1}|$. Hence, by a telescopic sum,

$$\left|\sum_{i=0}^{t} c_{a+2i}y_{a+2i}\right| \leq \sum_{i=0}^{t} |c_{a+2i}|r_{a+2i} = |c_{a-1}| - |c_{a+2t+1}|.$$

Consequently,

$$\begin{aligned}(-1)^e\sum_{m=e}^{f} c_m y_m &\geq |c_e| - \sum_{i=0}^{\lfloor(f-e-1)/2\rfloor} c_{e+1+2i}y_{e+1+2i} && \text{since } y_e \geq 1 \text{ and the sign parity of } c_m \text{ is } (-1)^m\\ &\geq |c_e| - (|c_e| - |c_{2\lfloor(f-e-1)/2\rfloor+e+2}|)\\ &= |c_{2\lfloor(f-e+1)/2\rfloor+e}|\end{aligned}$$

which proves the second part and the lower bound on $|\sum_{m=e}^{f} c_m y_m|$ in the first part of the claim. As the sign parity of $\sum_{m=e}^{f} c_m y_m$ is $(-1)^e$, we also have

$$\left|\sum_{m=e}^{f} c_m y_m\right| \leq \left|\sum_{i=0}^{\lfloor(f-e)/2\rfloor} c_{e+2i}y_{e+2i}\right| \leq |c_{e-1}| - |c_{2\lfloor(f-e)/2\rfloor+e+1}| \leq q - c_0.$$ □

5 Limitations on Improving the Bounds: Proof of Theorem 1.7

We will show here that the techniques of [21] cannot hope to prove much stronger lower bounds. We do this by constructing polynomials for which the lower bound we proved earlier is tight. We begin by showing this in the case of two different set sizes. We can normalize with respect to the bigger set size to assume that the weights are $-k$ and αk ($\alpha \in [0,1]$) without loss of generality. Clearly, $k \leq \log n$.

Lemma 5.1. *Let n, d, Δ be such that $d \leq n$. For any $\alpha \in [0,1]$ let $w \in \{-k, \alpha k\}^d$ be a word with $|w_{[d]}| \leq k$. There is a polynomial $P_\Delta \in \mathbb{F}_{sm}[\overline{X}(w)]$ which is computable by a set-multilinear formula of product-depth at most Δ, size at most $n^{O\left(\Delta d^{\mu(\Delta)}\right)}$ and has the maximum possible relative rank.*

Remark 5.2. We can replace αk with $\lfloor \alpha k \rfloor$ and assume that the weights in w are integers. It can be shown that this will not change the arguments in any significant way (see Claim 5.8).

We will need the extensive notation from [21], which we restate here.

5.1 Notation

- As in Section 2 and from the remark above, we assume $|X(w_i)| = 2^{|w_i|}$ and that the variables are indexed by binary strings $\{0,1\}^{|w_i|}$.
- Given any subset $S \subseteq [d]$, we denote by $S_+ = \{i \in S \mid w_i > 0\}$ the positive indices of S and similarly by S_-, the negative indices.
- We let $K = \sum_{i \in [d]} |w_i|$, $k_+ = \sum_{i \in S_+} |w_i|$ and $k_- = \sum_{i \in S_-} |w_i|$. We say S is $\mathcal{P}$-heavy if $k_+ \geq k_-$ and $\mathcal{N}$-heavy otherwise.
- Setting $I = [K]$, we partition the set $I = I_1 \cup \cdots \cup I_d$ where I_j is an interval of length $|w_j|$ that starts at $\left(\sum_{i<j} |w_j|\right) + 1$. Given a $T \subseteq [d]$, we let $I(T) = \bigcup_{j \in T} I_j$.
- Let $m = m_+ m_- \in \mathcal{M}_w^S$ be a monomial supported on variable sets indexed by S, with $m_+ \in \mathcal{M}_w^{S_+}$ and $m_- \in \mathcal{M}_w^{S_-}$. The Boolean string $\sigma(m_+)$ associated with the positive monomial (as defined in Section 2) can be thought of as a labeling of the elements of $I(S_+)$ in the natural way - $\sigma(m_+) : I(S_+) \longrightarrow \{0,1\}$. Similarly, for $\sigma(m_-)$.

Given a set S, we define a sequence of polynomials that we will later show to have set multilinear formulas of small size but large rank.

Fix $J_+ \subseteq I(S_+)$ and $J_- \subseteq I(S_-)$ such that $|J_+| = |J_-| = \min\{k_+, k_-\}$. Let π be a bijection from J_+ to J_-. Such a tuple (S, J_+, J_-, π) is called valid. Fix a valid (S, J_+, J_-, π).

A string $\tau \in \{0,1\}^{|k_+ - k_-|}$ defines a map $I(S_+) \backslash J_+ \longrightarrow \{0,1\}$ if S is $\mathcal{P}$-heavy and a map $I(S_-) \backslash J_- \longrightarrow \{0,1\}$ if S is $\mathcal{N}$-heavy.

The polynomial $P_{(S, J_+, J_-, \pi, \tau)}$ is the sum of all monomials m such that

(1) $\sigma(m_+)(j) = \sigma(m_-)(\pi(j))$ for all $j \in J_+$, and
(2) $\sigma(m_+)(j) = \tau(j)$ for all $j \in I(S_+) \setminus J_+$ if S is $\mathcal{P}$-heavy or $\sigma(m_-)(j) = \tau(j)$ for all $j \in I(S_-) \setminus J_-$ if S is $\mathcal{N}$-heavy.

As observed in [21], these polynomials have desirable properties that help build formulas for them inductively.

(P1) For any valid (S, J_+, J_-, π) and any $\tau \in \{0,1\}^{|k_+ - k_-|}$ the matrix $M_{w_{|S}}(P_{(S, J_+, J_-, \pi, \tau)})$ has the maximum possible rank for a matrix with its dimensions:

$$\operatorname{rank}(M_{w_{|S}}(P_{(S, J_+, J_-, \pi, \tau)})) = \min\{|\mathcal{M}_w^{S_+}|, |\mathcal{M}_w^{S_-}|\} = 2^{\min\{k_+, k_-\}}$$

(P2) Let $(S_i, J_{i,+}, J_{i,-}, \pi_i)$ ($i \in [r]$) be valid tuples with $S_i (i \in [r])$ being all $\mathcal{P}$-heavy and pairwise disjoint. Also assume that we have $\tau_i \in \{0,1\}^{k_{i,+} - k_{i,-}}$ where $k_{i,+} = \sum_{j \in I(S_{i,+})} w_j$. We can construct a new polynomial using these. Let $S = \bigcup_i S_i$ (also $\mathcal{P}$-heavy by definition), $J_+ =$

$\bigcup_i J_{i,+}$, $J_- = \bigcup_i J_{i,-}$, $\pi = \bigcup_i \pi_i$ and $\tau = \bigcup_i \tau_i$. Then, (S, J_+, J_-, π) is a valid tuple and moreover

$$P_{(S,J_+,J_-,\pi,\tau)} = \prod_{i=1}^{r} P_{(S_i,J_{i,+},J_{i,-},\pi_i,\tau_i)}$$

If each S_i is $\mathcal{N}$-heavy, an analogous fact can be shown to hold.

(P3) Say S', S'' are disjoint sets where S' is $\mathcal{P}$-heavy and S'' is $\mathcal{N}$-heavy. Also fix any valid (S', J'_+, J'_-, π') and $(S'', J''_+, J''_-, \pi'')$.

Assume that $S = S' \cup S''$ is $\mathcal{P}$-heavy. Let $J_- = I(S_-)$ and $J_+ = J'_+ \cup J''_+ \cup J'''$, where $J''' \subseteq I(S'_+)$ is any set of size $|I(S''_-)| - |I(S''_+)|$ disjoint from $J'_+ \cup J''_+$ (As S is $\mathcal{P}$-heavy, a set like this exists). Fix any bijection $\pi''' : J''' \to I(S''_-) \setminus J''_-$ Assume $\pi : J_+ \to J_-$ is defined to be $(\pi \cup \pi'' \cup \pi''')(j)$ for $j \in J'_+ \cup J''_+ \cup J'''$

Also, fix any $\tau : I(S_+) \setminus J_+ \to \{0, 1\}$. Any $\tau' : I(S'_+) \setminus J'_+ \to \{0, 1\}$ is said to extend τ if τ' restricts to τ on the set $I(S_+) \setminus J_+$ (note that J_+ contains $J''_+ = I(S''_+)$ and hence $I(S_+) \setminus J_+ \subseteq I(S'_+) \setminus J'_+$, so this definition makes sense). We denote by $\tau' \setminus \tau$ the restriction of τ' to the set J'''. We thus obtain

$$P_{(S,J_+,J_-,\pi,\tau)} = \sum_{\tau' \text{ extends } \tau} P_{(S',J'_+,J'_-,\pi',\tau')} \cdot P_{(S'',J''_+,J''_-,\pi'',(\tau'\setminus\tau)\circ\pi'''^{-1})}$$

The size of this sum is $2^{|J'''|} = 2^{k''_- - k''_+}$. An analogous identity holds when S is $\mathcal{N}$-heavy.

5.2 Cost of Building Formulas

To proceed, we introduce a few notions that help make the ideas in the proof overview (of Section 3.2) precise. We will only consider the case where $|w_{[d]}| \le k$, that is, $||\mathcal{P}_w|\alpha - |\mathcal{N}_w|| \le 1$. Fix Δ as in Lemma 5.1, and recall that $\lfloor r \rceil$ denotes the nearest integer to the real number r.

Definition 5.3 (Fractional Cost). Set $\mathrm{fc}(0) = 1$ and for $1 \le \delta \le \Delta - 1$,

$$\mathrm{fc}(\delta) := \min_{q < d^{\mu(\Delta)}/\mathrm{fc}(\delta-1)} |q\alpha - \lfloor q\alpha \rceil|/q,$$

where $q \in \mathbb{N}$ is a natural number. In case $\mathrm{fc}(\delta') = 0$ for some $\delta' \le \Delta - 1$, we set $\mathrm{fc}(\delta) = 0$ for all $\delta > \delta'$ as well. Let $\hat{\Delta} \le \Delta - 1$ be the largest integer such that $\mathrm{fc}(\hat{\Delta}) \ne 0$.

For $1 \le \delta \le \hat{\Delta}$, we denote by p_δ the (least) value of q for which the expression for $\mathrm{fc}(\delta)$ attains the minimum. Note that, by definition,

$$p_\delta \le d^{\mu(\Delta)}/\mathrm{fc}(\delta - 1). \tag{5.1}$$

We also denote by $n_\delta := \lfloor p_\delta \alpha \rceil$ the nearest integer to $p_\delta \alpha$. We first observe that the fractional cost falls exponentially with depth.

Claim 5.4 (Exponential Decline). *For $0 \le \delta \le \Delta - 1$,*

$$\mathrm{fc}(\delta) \le 1/(d^{\mu(\Delta)})^{F(\delta+1)-2}.$$

Proof of Claim 5.4. Note that when $\delta = 0$, the claim holds since $1 = \mathrm{fc}(0) \le 1/(d^{\mu(\Delta)})^{F(1)-2} = 1$. The claim also holds trivially if $\mathrm{fc}(\delta) = 0$. To prove the claim when $1 \le \delta \le \hat{\Delta}$, we will use Dirichlet's approximation principle ([29, Theorem 1A]), which essentially implies that for any real numbers α, and $N \ge 1$, there exists an integer $q \le N$ such that the distance from $q\alpha$ to the nearest integer is bounded by $1/N$. Consequently, we get that there exists an integer $q' \le d^{\mu(\Delta)}/\mathrm{fc}(\delta - 1)$ such that

$$|q'\alpha - \lfloor q'\alpha \rceil| < \mathrm{fc}(\delta - 1)/d^{\mu(\Delta)}. \tag{5.2}$$

When $\delta = 1$, the claim now follows from the definition since

$$\mathrm{fc}(1) = \min_{q < d^{\mu(\Delta)}/\mathrm{fc}(0)} |q\alpha - \lfloor q\alpha \rceil|/q \le |q'\alpha - \lfloor q'\alpha \rceil|/q' < 1/d^{\mu(\Delta)},$$

where we used the fact that $q' \ge 1$.

When $\delta \ge 2$, we claim that the q' obtained in Equation (5.2) isn't too small:

$$q' \ge d^{\mu(\Delta)}/\mathrm{fc}(\delta - 2). \tag{5.3}$$

Indeed, if not, then

$$\begin{aligned} \mathrm{fc}(\delta - 1) &= \min_{q < d^{\mu(\Delta)}/\mathrm{fc}(\delta-2)} \frac{|q\alpha - \lfloor q\alpha \rceil|}{q} \\ &\le \mathrm{fc}(\delta - 1)/d^{\mu(\Delta)}. && \text{from Equation (5.2), and } q' \text{ is now a candidate} \end{aligned}$$

This leads to a contradiction since $d^{\mu(\Delta)} > 1$. Hence Equation (5.3) holds, and we obtain the following bound on $\mathrm{fc}(\delta)$ using Equations (5.2) and (5.3):

$$\mathrm{fc}(\delta) \le \frac{|q'\alpha - \lfloor q'\alpha \rceil|}{q'} \le \frac{\mathrm{fc}(\delta - 1)}{d^{\mu(\Delta)}} \cdot \frac{\mathrm{fc}(\delta - 2)}{d^{\mu(\Delta)}}. \tag{5.4}$$

Solving Equation (5.4) readily gives

$$\mathrm{fc}(\delta) \le \frac{1}{d^{f(\delta)\mu(\Delta)}}, \qquad \text{where } f(i) \ge f(i-1) + f(i-2) + 2. \tag{5.5}$$

Rearranging, we have, $f(i) + 1 \ge (f(i-1) + 1) + (f(i-2) + 1) + 1$ whence we see that setting $f(i-1) + 1 := F(i) - 1$ satisfies the required constraints and proves the claim. □

We make an additional useful observation. Recall that $|\mathcal{P}_w|$ is the total number of positive sets of variables, and $|\mathcal{N}_w|$ is the total number of negative sets. Now, $\mu(\Delta) = \frac{1}{F(\Delta)-1}$ implies $\mu(\Delta)f(\Delta) = 1 + \mu(\Delta)$. Combining this with Claim 5.4, we get $\mathrm{fc}(\Delta - 1) \le 1/d^{1-\mu(\Delta)} = d^{\mu(\Delta)}/d$. Further noting that $d \ge |\mathcal{P}_w|$, we find

$$\mathrm{fc}(\Delta - 1) \le d^{\mu(\Delta)}/|\mathcal{P}_w|. \tag{5.6}$$

Let Δ' be the smallest integer for which $\mathrm{fc}(\Delta') \le d^{\mu(\Delta)}/|\mathcal{P}_w|$ holds (such a Δ' exists and is bounded above by $\Delta - 1$ from Equation (5.6)). In fact, note that $\Delta' \le \hat{\Delta} + 1$ since $\mathrm{fc}(\hat{\Delta} + 1) = 0$. We will now (re)define $p_{\Delta'+1} := |\mathcal{P}_w|$ and $n_{\Delta'+1} := |\mathcal{N}_w|$.

With the notation in place, we can now state the following central claim that constructs the polynomial needed for Lemma 5.1:

Claim 5.5. *Let Δ, Δ' be as fixed above. For any integer $\delta \le \Delta' + 1$, if $S \subseteq [d]$ satisfies $|w_S| \le k$, $|S_+| \le p_\delta$ and $|S_-| \le n_\delta$, then there exist J_+, J_-, π such that (S, J_+, J_-, π) is valid and for all $\tau \in \{0,1\}^{|k_+ - k_-|}$, the polynomial $P_{(S, J_+, J_-, \pi, \tau)}$ can be computed by a set-multilinear formula of product-depth δ and size at most $|S|^\delta 2^{5k\delta d^{\mu(\Delta)}}$.*

We finish the proof of Lemma 5.1 assuming the above claim:

Proof of Lemma 5.1. We know that $|w_{[d]}| \le k$. Recall that $p_{\Delta'+1} = |\mathcal{P}_w|$ and $n_{\Delta'+1} = |\mathcal{N}_w|$. We can now apply Claim 5.5 with $S = [d]$ and $\delta = \Delta' + 1$. This gives a polynomial $P_{\Delta'+1} \in \mathbb{F}_{sm}[\overline{X}(w)]$ with $\mathrm{relrk}_w(P_{\Delta'+1}) = 2^{-|w_{[d]}|/2}$. The polynomial $P_{\Delta'+1}$ is computable by a set-multilinear formula of product-depth at most $\Delta' + 1 \le \Delta$, and size at most $d^\Delta 2^{5k\Delta d^{\mu(\Delta)}} \le n^{O\left(\Delta d^{\mu(\Delta)}\right)}$. □

The following claim is the main technical result that helps prove Claim 5.5. It is in the same spirit as [21, Claim 28], but we show the existence of a better partition with a more careful analysis. Furthermore, our analysis holds for any $\alpha \in [0, 1]$.

Claim 5.6. *Fix* $1 \leq \delta \leq \Delta' + 1$. *Let* $S \subseteq [d]$ *with* $|w_S| \leq k$ *such that* $|S_+| \leq p_\delta$ *and* $|S_-| \leq n_\delta$. *Then there exists a partition of S as* $S_1 \cup S_2 \cup \ldots S_r$, *where the following conditions hold:*

(1) $|S_{i,+}| \leq p_{\delta-1}$ *and* $|S_{i,-}| \leq n_{\delta-1}$ *for all* $i \in [r]$.
(2) $\sum_{i=1}^{r} |w_{S_i}| \leq 5kd^{\mu(\Delta)}$.
(3) $|w_{S_i}| \leq k$ *for all* $i \in [r]$.

Proof of Claim 5.6. As long as possible, pick sets S_i with $|S_{i,+}| = p_{\delta-1}$ positive indices and $|S_{i,-}| = n_{\delta-1}$ negative indices. For all such sets picked, we have

$$|w_{S_i}| = \left|\sum\nolimits_{j \in S_i} w_j\right| = k \cdot |p_{\delta-1}\alpha - n_{\delta-1}| \leq k. \tag{5.7}$$

Suppose the sets chosen after the procedure are $S_1, \ldots, S_m$, where $m = \min\{\lfloor \frac{|S_+|}{p_{\delta-1}} \rfloor, \lfloor \frac{|S_-|}{n_{\delta-1}} \rfloor\}$ and we are left with the set S'. Since we cannot pick the sets anymore, we must have that $|S'_+| < p_{\delta-1}$ or $|S'_-| < n_{\delta-1}$ (or both). We will have to deal with two cases to pick the next sets.

Case 1: $\alpha|S'_+| \leq |S'_-|$.

We pick a set S_{m+1} with $|S'_+|$ positive indices and $b = \lfloor \alpha |S'_+| \rceil$ negative indices (notice that $b \leq |S'_-|$) so that

$$|w_{S_{m+1}}| = k\left|\alpha|S'_+| - b\right| \leq k. \tag{5.8}$$

Note that if $|S'_+| > p_{\delta-1}$, then $|S'_-| \geq \lfloor \alpha|S'_+| \rceil \geq \lfloor \alpha p_{\delta-1} \rceil = n_{\delta-1}$ which contradicts with the fact that either $|S'_+| < p_{\delta-1}$ or $|S'_-| < n_{\delta-1}$. Therefore, we have $|S'_+| \leq p_{\delta-1}$ and $b = \lfloor \alpha|S'_+| \rceil \leq \lfloor \alpha p_{\delta-1} \rceil = n_{\delta-1}$ which ensures that condition (1) is satisfied for $i = m + 1$.

The remaining set $T = S' \setminus S_{m+1}$ has only negative values which we split into singletons $S_{m+2}, \ldots, S_r$ (there are $(|S'_-| - b)$ of these sets). As these are singletons, for $m + 2 \leq j \leq r$, we trivially have $|w_{S_j}| \leq k$.

We also note that

$$\begin{aligned}
|S'_-| - b &= (|S'_-| - \alpha|S'_+|) + (\alpha|S'_+| - b) \\
&= (|S_-| - m \cdot n_{\delta-1} - \alpha|S_+| + \alpha m \cdot p_{\delta-1}) + (\alpha|S'_+| - b) \\
&\leq ||S_-| - \alpha|S_+|| + m|p_{\delta-1}\alpha - n_{\delta-1}| + |\alpha|S'_+| - b|,
\end{aligned}$$

where the first term is at most 1 since $|w_S| \leq k$ and the last term is at most 1 as well. Putting it all together,

$$\sum_{i=m+2}^{r} |w_{S_i}| = (|S'_-| - b)k \leq (m|p_{\delta-1}\alpha - n_{\delta-1}| + 2)k.$$

Case 2: $\alpha|S'_+| > |S'_-|$.

Observe that if $|S'_-| > n_{\delta-1}$, then we must have $|S'_+| < p_{\delta-1}$ implying $|S'_-| \leq \lfloor \alpha|S'_+| \rceil \leq \lfloor \alpha p_{\delta-1} \rceil = n_{\delta-1}$. This is a contradiction. Therefore, we have $|S'_-| \leq n_{\delta-1}$.

Since $|S'_-| \leq n_{\delta-1} = \lfloor \alpha p_{\delta-1} \rceil$ and $|S'_-| \leq \lfloor \alpha|S'_+| \rceil$, there exists $c \leq \min(p_{\delta-1}, |S'_+|)$ such that $\lfloor \alpha c \rceil = |S'_-|$. Pick a set S_{m+1} with $|S'_-|$ negative indices and c positive indices so that

$$|w_{S_{m+1}}| = k|\alpha c - |S'_-|| \leq k.$$

Condition (1) is clearly satisfied for $i = m + 1$.

The remaining set $T = S' \setminus S_{m+1}$ has only positive values which we split into singletons $S_{m+2}, \ldots, S_r$ (there are $(|S'_+| - c)$ of these sets). As these are singletons, for $m + 2 \leq j \leq r$, we trivially have $|w_{S_j}| \leq k$.

Similar to the earlier case,

$$\begin{aligned}\sum_{i=m+2}^{r} |w_{S_i}| &= (|S'_+| - c)\alpha k \\ &= \left((\alpha|S'_+| - |S'_-|) + (|S'_-| - \alpha c)\right) k \\ &\leq \left(|\alpha|S_+| - |S_-|| + m|p_{\delta-1}\alpha - n_{\delta-1}| + ||S'_-| - \alpha c|\right) k \\ &\leq (m|p_{\delta-1}\alpha - n_{\delta-1}| + 2)k.\end{aligned}$$

Therefore, in both of the above cases,

$$\begin{aligned}\sum_{i=1}^{r} |w_{S_i}| &= \sum_{i=1}^{m} |w_{S_i}| + |w_{S_{m+1}}| + \sum_{i=m+2}^{r} |w_{S_i}| \\ &\leq km|p_{\delta-1}\alpha - n_{\delta-1}| + k + (m|p_{\delta-1}\alpha - n_{\delta-1}| + 2)k \\ &\leq k\left(2\left\lfloor\frac{|S_+|}{p_{\delta-1}}\right\rfloor |p_{\delta-1}\alpha - n_{\delta-1}| + 3\right) \leq k\left(2|S_+|\frac{|p_{\delta-1}\alpha - n_{\delta-1}|}{p_{\delta-1}} + 3\right) \\ &\leq k\left(2p_\delta \cdot \mathrm{fc}(\delta-1) + 3\right) \qquad \text{(By definition of fc)} \\ &\leq 5kd^{\mu(\Delta)}\end{aligned}$$

where the last inequality is true because $\mathrm{fc}(\delta-1) \leq d^{\mu(\Delta)}/p_\delta$ holds for $\delta \leq \Delta'$ from Equation (5.1); it also holds for $\delta = \Delta' + 1$ by the definition of Δ'. □

Armed with all this, the proof of Claim 5.5 becomes quite similar to the proof of Claim 27 in [21].

Proof of Claim 5.5. The proof is by induction on the product-depth δ for all $1 \leq \delta \leq \Delta' + 1$, where $\Delta' + 1$ is as defined above.

- **Base case**: When $\delta = 1$, we use the trivial expression for $P_{(S,J_+,J_-,\pi,\tau)}$ as a sum of monomials. This is a product-depth one $\sum\prod$ set-multilinear formula of size at most $2^{k|S|} \leq 2^{k(p_1+n_1)}$. Note that since $|w_S| \leq k$, $|p_1\alpha k - n_1 k| \leq k$. This gives $n_1 \leq p_1\alpha + 1 \leq p_1 + 1$. Using the bound $p_1 \leq d^{\mu(\Delta)}$ from Equation (5.1), we obtain a size bound of $2^{2k(d^{\mu(\Delta)}+1)} \leq |S|2^{5kd^{\mu(\Delta)}}$, as required.
- **Induction step**: Consider some $\delta > 1$. Let $k_+ := |I(S_+)|$ and $k_- := |I(S_-)|$. Without loss of generality, we can assume S is $\mathcal{P}$-heavy. We know that $|w_S| \leq k$, $|S_+| \leq p_\delta$ and $|S_-| \leq n_\delta$. Thus, using Claim 5.6, we obtain a partition of $S = S_1 \cup \cdots \cup S_r$, where for all $i \in [r]$, we have $|w_{S_i}| \leq k, |S_{i,+}| \leq p_{\delta-1}, |S_{i,-}| \leq n_{\delta-1}$ and

$$\sum_{i=1}^{r} |w_{S_i}| \leq 5kd^{\mu(\Delta)}. \tag{5.9}$$

  By induction hypothesis, for every $i \in [r]$, there exist $J_{i,+}, J_{i,-}, \pi_i$ such that $(S_i, J_{i,+}, J_{i,-}, \pi_i)$ is valid and for each $\tau_i \in \{0,1\}^{|k_{i,+}-k_{i,-}|}$, the polynomial $P_{(S_i,J_{i,+},J_{i,-},\pi_i,\tau_i)}$ has a set-multilinear formula F_{i,τ_i} of product-depth $\delta - 1$ and size $s_i \leq |S_i|^{\delta-1}2^{5k(\delta-1)d^{\mu(\Delta)}}$.

  We can assume that $S_1, \ldots, S_\gamma$ are $\mathcal{P}$-heavy and $S_{\gamma+1}, \ldots, S_r$ are $\mathcal{N}$-heavy, where $\gamma \in [r]$. Using (P2) above, we get that

$$P_{(S',J'_+,J'_-,\pi',\tau')} = \prod_{i=1}^{\gamma} P_{(S_i,J_{i,+},J_{i,-},\pi_i,\tau_i)}\,,\quad P_{(S'',J''_+,J''_-,\pi'',\tau'')} = \prod_{i=\gamma+1}^{r} P_{(S_i,J_{i,+},J_{i,-},\pi_i,\tau_i)} \tag{5.10}$$

where

$$(S', J'_+, J'_-, \pi') = \left(\bigcup_{i\in[\gamma]} S_i, \bigcup_{i\in[\gamma]} J_{i,+}, \bigcup_{i\in[\gamma]} J_{i,-}, \bigcup_{i\in[\gamma]} \pi_i \right),$$
$$(S'', J''_+, J''_-, \pi'') = \left(\bigcup_{i=\gamma+1}^{r} S_i, \bigcup_{i=\gamma+1}^{r} J_{i,+}, \bigcup_{i=\gamma+1}^{r} J_{i,-}, \bigcup_{i=\gamma+1}^{r} \pi_i \right)$$

and for $i \in [\gamma]$, each τ_i is a restriction of τ' to $I(S_{i,+}) \setminus J_{i,+}$ whereas for $i \in \{\gamma+1, \ldots, r\}$, each τ_i is a restriction of τ'' to $I(S_{i,-}) \setminus J_{i,+}$.

Note that both these tuples are valid and S' is $\mathcal{P}$-heavy and S'' is $\mathcal{N}$-heavy. Then, using (P3), we construct the polynomial

$$\begin{aligned} P_{(S,J_+,J_-,\pi,\tau)} &= \sum_{\tau' \text{ extends } \tau} P_{(S',J'_+,J'_-,\pi',\tau')} \cdot P_{(S'',J''_+,J''_-,\pi'',\tau'')} \\ &= \sum_{\tau' \text{ extends } \tau} \prod_{i=1}^{r} P_{(S_i,J_{i,+},J_{i,-},\pi_i,\tau_i)} \end{aligned}, \tag{5.11}$$

where (S', J'_+, J'_-, π') and $(S'', J''_+, J''_-, \pi'')$ are constructed as in (P3). We can now use the formulas F_{i,τ_i} we had before from induction and construct a set-multilinear product-depth δ formula for $P_{(S,J_+,J_-,\pi,\tau)}$ of size at most

$$\begin{aligned} r \cdot 2^{|k''_- - k''_+|} \cdot \max_{i\in[r]} s_i &\le |S| \cdot 2^{\sum_i |w_{S_i}|} \cdot |S_i|^{\delta-1} 2^{5k(\delta-1)d^{\mu(\Delta)}} \\ &\le |S| \cdot 2^{5kd^{\mu(\Delta)}} \cdot |S|^{\delta-1} 2^{5k(\delta-1)d^{\mu(\Delta)}} \\ &\le |S|^{\delta} 2^{5k\delta d^{\mu(\Delta)}} \end{aligned}, \tag{5.12}$$

where the second inequality follows from Equation (5.9). □

5.3 Handling More than Two Weights

With multiple weights, we partition the index set $[d]$ into sets $\{S_i\}$ such that the sub-word indexed by each S_i contains at most two distinct weights. This allows us to reduce the case of multiple weights to that of two weights, for which the machinery we built in the previous section can be used to prove upper bounds. We start by describing this reduction.

LEMMA 5.7. *Let $w \in \{\alpha_1, \ldots, \alpha_\gamma\}^d$ ($|\alpha_i| \le k$ for all i) be a word with $\gamma \le d$ different weights and $|w_{[d]}| \le k$. Then, the index set $[d]$ can be partitioned as $S_1 \cup \cdots \cup S_\eta$ with $\eta \le 6\gamma$ such that for all $i \in [\eta]$, the sub-word $w_{|S_i}$ has at most two distinct weights and further, $|w_{S_i}| \le k$.*

PROOF. Let $\{T_1, \ldots, T_\gamma\}$ be a partition of $[d]$, where every set T_j in the partition corresponds to one weight (i.e., for every $I \in T_j$, $w_i = \alpha_j$). We give an algorithm to obtain the desired partition of $[d]$. The basic idea is to take two distinct weights and group as many buckets corresponding to these set sizes as possible while maintaining the constraint on the sum of weights.

(1) Initialize $j = 1$ and $\pi := \{T_1, \ldots, T_\gamma\}$. Repeat the following steps until π is empty.
(2) If possible, pick sets T_p and T_n from π such that α_p is positive and α_n is negative.
(3) If $|T_p|\alpha_p + |T_n|\alpha_n \le 0$, then it is easy to see that we can pick a subset $T'_n \subseteq T_n$ such that $||T_p|\alpha_p + |T'_n|\alpha_n| \le k$ since $|\alpha_p|, |\alpha_n| \le k$.
(4) Set $S_j := T_p \cup T'_n$. We have $|w_{S_j}| = ||T_p|\alpha_p + |T'_n|\alpha_n| \le k$ as required. Set $T_n := T_n \setminus T'_n$. Drop T_p from π. If $|T_p|\alpha_p + |T_n|\alpha_n \ge 0$, proceed in a similar way.

(5) If we can not pick two sets T_p and T_n as above, it means that for the remaining sets in π, either their corresponding weights are all positive or all negative. We consider the case when they are all positive (the other case can be dealt with analogously).
   (a) If there exists a set T_p such that $|T_p|\alpha_p \leq k$, then set $S_j := T_p$ and drop T_p from π.
   (b) Otherwise, consider any remaining set T_p. We have $|T_p|\alpha_p > k$. Since $\alpha_p \leq k$, there exist $T'_p \subseteq T'_p \cup \{q\} \subseteq T_p$ such that $|T'_p|\alpha_p \leq k$ and $(|T'_p| + 1)\alpha_p > k$. Set $S_j := T'_p$, $S_{j+1} = \{q\}$ and $T_p := T_p \setminus (T'_p \cup \{q\})$. Increment $j = j + 1$.

(6) Increment $j = j + 1$ and continue.

We have ensured that $|w_{S_i}| \leq k$ for all i. It suffices to show that the steps 2-6 are repeated at most 3γ times. Every time step 4 or step $5.a$ is executed, the size of π reduces by at least 1. Hence, they can be repeated at most γ times in total. When step $5.b$ is executed for the first time, we know that the remaining collection of sets is $\pi = \{T_1, \ldots, T_\beta\}$, where each T_j corresponds to a positive weight. Let us denote the weight of this collection by $w_\pi = \sum_{j=1}^{\beta} w_{T_j} = \sum_{j=1}^{\beta} |T_j|\alpha_j$. Suppose till now we have picked the sets $S_1, \ldots, S_{\beta'}$ for some $\beta' \leq \gamma$. Then $w_\pi = w_S - \sum_{i=1}^{\beta'} w_{S_i}$. Using the triangle inequality, $w_\pi \leq |w_S| + \sum_{i=1}^{\beta'} |w_{S_i}| \leq k + \gamma k$. Every time we remove two sets $S_j = T'_p$ and $S_{j+1} = \{q\}$ as in step $5.b$, the value of w_π reduces by $(|T'_p| + 1)\alpha_p > k$. Hence, this can be repeated at most $\gamma + 1$ times. □

We can now construct polynomials with small set-multilinear formula size but large rank, even when the number of distinct set sizes is not two. We construct two different polynomials that are useful in different regimes of the number of set sizes (see Remark 1.8).

Proof of Theorem 1.7. As $|w_{[d]}| \leq k$, by Lemma 5.7, we get a partition of the index set $[d]$ into sets $S_1, \ldots, S_\eta$ ($\eta \leq 6\gamma$) such that the sub-word corresponding to each $S_i \subseteq [d]$ contains at most two weights and $|w_{S_i}| \leq k$.

— **Constructing** P_Δ: Coresponding to each set S_i, we have a size parameter α_i and the corresponding fractional cost function fc_i. As in Section 5.2, we also have a $\Delta'_i \leq \Delta - 1$ and sequences $\{p^i_\delta\}_{\delta=0}^{\Delta-1}$ and $\{n^i_\delta\}_{\delta=0}^{\Delta-1}$ with $p_{\Delta'_i+1} = |S_{i,+}|$ and $n_{\Delta'_i+1} = |S_{i,-}|$.

  We apply Claim 5.6 to each S_i to get a partition $S_i = S_{i1} \cup \cdots \cup S_{ir_i}$, with $|S_{ij,+}| \leq p_{\Delta'_i}$, $|S_{ij,-}| \leq n_{\Delta'_i}$ and $|w_{S_{ij}}| \leq k$ for all $j \in [r_i]$. Moreover, $\sum_{j\in[r_i]} |w_{S_{ij}}| \leq 5k|S_i|^{\mu(\Delta)}$. Applying Claim 5.5 to each of the S_{ij}s, we get that there exist $J_{ij,+}, J_{ij,-}, \pi_{ij}$ such that $(S_{ij}, J_{ij,+}, J_{ij,-}, \pi_{ij})$ is valid, and for all $\tau_{ij} \in \{0,1\}^{|k_+ - k_-|}$, the polynomial $P_{(S_{ij}, J_{ij,+}, J_{ij,-}, \pi_{ij}, \tau_{ij})}$ can be computed by a set-multilinear formula of depth at most $\Delta - 1$ (all the Δ'_is are at most $\Delta - 1$) and size

$$s_{ij} \leq |S_{ij}|^{\Delta-1} 2^{5k(\Delta-1)|S_i|^{\mu(\Delta)}} \leq |S_{ij}|^{\Delta-1} 2^{5k(\Delta-1)d^{\mu(\Delta)}}.$$

  We club all the $\mathcal{P}$-heavy sets together, and all the $\mathcal{N}$-heavy sets together across *all* the S_is. Now, using the exact same construction (which we skip) as in the induction part of the proof of Claim 5.5, we obtain a polynomial P_Δ of product-depth at most Δ and size at most

$$\sum_i r_i \cdot 2^{k''_- - k''_+} \cdot \max_{i\in[\eta], j\in[r_i]} s_{ij} \leq d \cdot 2^{\sum_{i\in[\eta], j\in[r_i]} |w_{S_{i,j}}|} \cdot d^{\Delta-1} 2^{5k(\Delta-1)d^{\mu(\Delta)}}$$
$$\leq d^\Delta 2^{30k\gamma\Delta d^{\mu(\Delta)}} \leq n^{O\left(\gamma\Delta d^{\mu(\Delta)}\right)}.$$

— **Constructing** Q_Δ: Once we have the sets $S_1 \ldots, S_\eta$ with $|w_{S_i}| \leq k$, we can apply Lemma 5.1 directly to each of these S_is where we set the product-depth to $\Delta - 1$. For all $i \in [\eta]$, we obtain polynomials $P_{(S_i, J_{i,+}, J_{i,-}, \pi_i, \tau_i)}$ with formulas of size

$$|S_i|^{\Delta-1} 2^{5k(\Delta-1)d^{\mu(\Delta-1)}},$$

  and product-depth $\Delta - 1$.

Again using the same construction as in the proof of Claim 5.5, we obtain the polynomial Q_Δ of product-depth Δ and size at most

$$\eta \cdot 2^{k''_- - k''_+} \cdot \max_{i\in[\eta]} s_i \le d \cdot 2^{\sum_{i\in[\eta]} |w_{S_i}|} \cdot d^{\Delta-1} 2^{5k(\Delta-1)d^{\mu(\Delta-1)}}$$
$$\le d^{\Delta} 2^{5k(\Delta-1)d^{\mu(\Delta-1)}+6\gamma k} \le n^{O\left(\Delta d^{\mu(\Delta-1)}+\gamma\right)}.$$

Note that by the properties described earlier, both these polynomials have the maximum possible relative rank $\mathrm{relrk}_w(P_\Delta) = \mathrm{relrk}_w(Q_\Delta) = 2^{-|w_{[d]}|/2}$. □

Finally, we show that in the above proofs, without loss of generality, it can be assumed that all entries of w are integers. We can always consider a word w' with integer entries such that the small-sized formula maximizing the relative rank for w' also nearly maximizes it for w, by which we mean that it differs from the maximum attainable relative rank by at most a factor of 2^d, which is not much since $d = o(\log n)$. We formalize this now.

Claim 5.8. *Let $S \subseteq [d]$ and let $w \in \{\alpha_1 k, \ldots, \alpha_\gamma k, -\beta_1 k, \ldots, -\beta_{\gamma'} k\}^d$,$(|\alpha_i|, |\beta_i| \le 1$ for all $i)$ be a word with $\gamma \le d$ different weights. Consider the word w', where every $\alpha_i k$ of w is replaced by $\lfloor \alpha_i k \rfloor$ and every $-\beta_j k$ of w is replaced by $-\lfloor \beta_j k \rfloor$. Let P' be the polynomial obtained in the proof of Theorem 1.7 for the word w'. Then,* $\mathrm{relrk}_w(P') \ge 2^{-d} 2^{-|w_{[d]}|/2}$.

Proof. From the definition of w', we have $|w'_i| \le |w_i| \le |w'_i| + 1$. Hence $\sum_i (|w_i| - |w'_i|) \le d$. Using the definition of relative rank and noting that $\mathrm{rank}(\mathcal{M}_w(P')) = \mathrm{rank}(\mathcal{M}_{w'}(P'))$,

$$\mathrm{relrk}_w(P')/\mathrm{relrk}_{w'}(P') = \ = \frac{1}{2^{\sum_i (|w_i| - |w'_i|)/2}} \ge 2^{-d/2}.$$

As P' is the polynomial obtained in Theorem 1.7 for the word w', we have

$$\mathrm{relrk}_{w'}(P') = 2^{-|w'_{[d]}|/2}.$$

Thus it suffices to show that $|w'_{[d]}| \le |w_{[d]}| + d$.

By the triangle inequality, $|\sum_i w'_i| \le |\sum_i w_i| + |\sum_i w'_i - w_i|$ which implies

$$|w'_{[d]}| \le |w_{[d]}| + \left|\sum_i w_i - w'_i\right| \le |w_{[d]}| + \sum_i |w_i| - |w'_i| \le |w_{[d]}| + d$$

where the second inequality holds because $|w_i| \ge |w'_i|$ for all i. □

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